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THE METHOD OF STRESS VARIATION APPLIED TO MINIMAL DESIGN FOR MULTIPLE LOADING

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Abstract—The method of stress variation, previously developed for the optimal design of axisymmetric sandwich plates obeying the Tresca criterion for a single loading condition[1–3], is extended to multiple loading conditions. The method consists of a systematic reduction in the design parameter until the optimum is reached. For the class of loadings treated in this paper, it is shown that repeated application of only a single stress variation is necessary to produce the optimum design for a simply supported plate. The method is such that it can easily be adapted to automatic computation, and the computer time required is simply proportional to the number of separate loads to be considered. An example of a simply supported plate for three separate loads is presented.

1. INTRODUCTION

The problem of designing minimum weight plastic structures that must carry different loads at different times (multiple loading) has been receiving increasing attention in recent years. Beam design for multiple loading conditions has been treated analytically by Mayeda and Prager[4] and Nagtegaal[5]; for moving loads (infinitely many different loads) by Gross and Prager[6] and Save and Prager[7].

In his treatment of sandwich plates, Shield[8], using the upper bound theorem of limit analysis, showed that a sufficient condition for optimality is

$$\sum_{i=1}^{n} D_i = M_0, \tag{1.1}$$

where D_i is the energy dissipation of the plate under the load p_i, n is the total number of different loads and M_0 is the yield moment of the minimum weight plate. Equation (1.1) has been generalized to the case where $n = \infty$ by Save and Shield[9], and a solution was presented for a certain class of moving loads on a simply supported plate.

Equation (1.1), together with the usual field equations for limit analysis, determines the optimal design as well as the optimal stresses and collapse mechanisms associated with each load p_i. When Tresca yield condition is adopted, the procedure generally entails the assumption of a certain collapse mechanism for each load condition p_i, which is then tested by (1.1) for kinematic admissibility† and by the equilibrium requirements for statical admissibility. When the collapse mechanisms for each load are particularly simple, the foregoing procedure is rather efficient; unfortunately however, the appropriate collapse mechanisms are often not of a simple nature, and the problem of assuming the correct collapse mechanism for each load p_i becomes a formidable task.

The purpose of this paper is to present an alternative procedure which avoids the necessity of assuming the correct collapse mechanisms since the procedure is based on statics alone. The method—the technique of stress variation—has previously proven itself sufficiently powerful to determine the minimal designs in all cases of full or annular axisymmetric sandwich plates for single uni-directional loads and any combination of edge supports[1–3]. The technique, as developed in this paper, does not divorce itself from (1.1); rather it complements the optimality criterion (1.1) by a logical and systematical development of designs whose optimality may then be tested against (1.1).

†For example, if $n = 2$, the assumption that the stresses corresponding to one load are at a corner of the Tresca condition, and those for the other load on a side yield three equations for the four principal curvature rates. The fourth equation is provided by (1.1). If the resulting solution yields curvature rates compatible with flow law, the assumed collapse mechanism is said to be kinematically admissible.

2. STATICAL FORMULATION

Consider an axisymmetric sandwich plate with prescribed kinematic support conditions. The fact sheets, separated by a core of constant depth, have a common thickness which depends on the radial coordinate r, and are assumed to fail in accordance with the Tresca yield criterion. The bending moments M_r^i, M_θ^i and shear force T_i associated with the load p_i are said to be admissible whenever they meet the conditions of equilibrium

$$\frac{d}{dr}(rM_r^i) - M_\theta^i = rT_i, \qquad \frac{d}{dr}(rT_i) = -rp_i, \qquad (i = 1, \ldots, n) \tag{2.1}$$

and statical boundary conditions. For each set of admissible stresses $(M_r^i, M_\theta^i; T_i)$ define the function M_0^i by

$$M_0^i = \max\{|M_r^i|, |M_\theta^i|, |M_r^i - M_\theta^i|\} \qquad (i = 1, \ldots, n). \tag{2.2}$$

Having adopted the Tresca criterion (Fig. 1), it follows from (2.2) that the plate will be at or below collapse if

$$M_0^i \leq M_0 \qquad (i = 1, \ldots, n). \tag{2,3}$$

Since the face sheet thickness t is proportional to the yield moment M_0, it suffices to treat M_0 as the design variable. With the definition of the class of safe designs,

$$M_0(r) \equiv \max_i M_0^i(r) \tag{2.4}$$

and homogeneity of the face sheets, minimizing the weight becomes equivalent to minimizing the moment volume J, where

$$J = \int_a^b rM_0(r)\,dr \tag{2.5}$$

over the class of admissible stresses, and where a and b are, respectively, the radii at the inner ($a = 0$ for the full plate) and outer edges of the plate. Equation (1.1) is a necessary and sufficient condition for J to be an absolute minimum; the proof is found in the appendix.

3. TECHNIQUE OF STRESS VARIATION

Suppose that the plate has k circles of support at the radii $d_j (j = 1, \ldots, k)$. Furthermore, denote the reaction at the support d_j, when the load p_i is applied, by R_j^i. Then according to (2.1), the shear force must be given by

$$rT_i(r) = -\int_a^r \xi p_i(\xi)\,d\xi + \sum_{j=1}^k d_j R_j^i H(r - d_j), \tag{3.1}$$

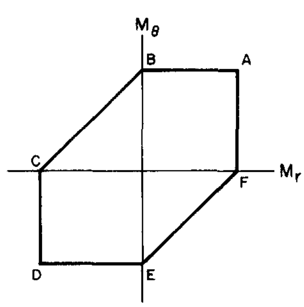

Fig. 1. Tresca yield criterion.

where $H(r-d_j)$ is the Heaviside operator with a source at $r=d_j$. The k reactions $R_j^i(j=1,\ldots,k)$ are related by only one force equation of equilibrium for each i. This equation may be written in the form

$$\sum_{j=1}^{k} d_j R_j^i = \beta_i, \tag{3.2}$$

where β_i is a known constant.

Now let $(M_r^i, M_\theta^i; T_i)$ be a set of admissible stresses corresponding to the load p_i. According to (2.1), (3.1) and (3.2), any other set of stresses $(M_r^i+\Delta M_r^i, M_\theta^i+\Delta M_\theta^i; T_i+\Delta T_i)$ will also be admissible if

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}(r\Delta M_r^i) &= \Delta M_\theta^i + r\Delta T_i, \\ r\Delta T_i &= \sum_{j=1}^{k} \rho_j^i d_j H(r-d_j), \\ 0 &= \sum_{j=1}^{k} \rho_j^i d_j, \end{aligned} \tag{3.3}$$

where ρ_j^i are arbitrary constants. The functions $(\Delta M_r^i, \Delta M_\theta^i; \Delta T_i)$ are called the stress variation; the stress variation is said to be admissible whenever it satisfies (3.3).

The stress variation technique consists in first choosing an initial admissible stress field and a corresponding design M_0 chosen in accordance with (2.2) and (2.4). The piecewise linearity of the Tresca criterion is exploited by a judicious choice of an admissible stress variation which, if sufficiently small, will render the change in moment volume independent of the initial admissible stresses. If the moment volume variation is negative, the process is repeated anew, with the resulting admissible stress taken as the new admissible stress. The design finally obtained by repeating the process until there is no further reduction in moment volume is then tested by (1.1) for absolute optimality.

All the results of the present study are obtained by using a single stress variation, namely

$$\begin{array}{ll} \Delta M_r^i = \Delta M_\theta^i = 0 & d^* < r \le b \\ \left.\begin{array}{l} \Delta M_\theta^i = \xi_i \\ \Delta M_r^i = \xi_i(r-d^*)/r \end{array}\right\} & d^*-\epsilon \le r \le d^* \\ \Delta M_r^i = \Delta M_\theta^i = -\xi_i\epsilon/(d^*-\epsilon) & a \le r < d^*-\epsilon \\ \Delta T = 0 & a \le r \le b. \end{array} \tag{3.4}$$

The effect of the variation (3.4) is depicted to Figs. 2, where the solid lines represent typical admissible moments (M_r^i, M_θ^i) and the dashed lines show the resulting admissible moments after application of the variation (3.4). In Fig. 2a it is assumed that $\xi_i>0$, while in Fig. 2b, $\xi_i<0$. Thus the variation (3.4) is characterized by an isotropic $(\Delta M_r^i = \Delta M_\theta^i)$ constant variation in the region $a \le r < d^*-\epsilon$, and a pulse variation $(\Delta M_\theta^i = \text{constant})$ in the region $d^*-\epsilon \le r \le d^*$. The stresses are left unchanged in the outer region $d^* < r \le b$.

To be specific, consider the simply supported full plate with radius $r=b$. If all the loadings $p_i(i=1,\ldots,n)$ are downward, then the statically determinate shear force T_i will be non-positive. A convenient initial admissible stress field is the isotropic field

$$M_r^i = M_\theta^i = \int_b^r T_i(\xi)\,\mathrm{d}\xi \qquad (i=1,\ldots,n). \tag{3.5}$$

Fig. 2. Stress variation. (a) $\xi_i>0$; (b) $\xi_i<0$.

For the case of three loads, each of which influence the design, assume the initial stress (3.5) is of the form sketched on Fig. 3a. According to (2.4) and Fig. 3a, the initial design is determined by

$$M_0 = \begin{cases} M_\theta^1 & 0 \le r \le r_1 \\ M_\theta^2 & r_1 \le r \le r_2 \\ M_\theta^3 & r_2 \le r \le b. \end{cases} \tag{3.6}$$

Now apply the stress variation (3.4) where d^* satisfies $r_2 \le d^* \le b$ but is otherwise arbitrary, and where $\xi_i = M_r^3(d^*) - M_r^i(d^*) \ge 0$ and ϵ is an infinitesimal quantity. The variation clearly raises the M_θ^i diagram to be coincident with the yield moment on the infinitesimal interval $d^* - \epsilon \le r \le d^*$, but it maintains isotropy in the region $r < d^* - \epsilon$ by lowering both M_θ^i and M_r^i by an equal amount. The new admissible stresses are shown in Fig. 3b. Note that all the stress states, both before and after the variation, lie either on side AB or at corner A of the Tresca condition (Fig. 1). Consequently the new design is still determined by (3.6) where the M_θ^i are now the moments after application of the variation. The moment volume variation may now be easily calculated in terms of the stress variation (3.4), viz.

$$\Delta J = \int_0^{r_1} \Delta M_\theta^1 r \, dr + \int_{r_1}^{r_2} \Delta M_\theta^2 r \, dr = -[\xi_1 r_1 + \xi_2(r_2 - r_1)]\epsilon / d^* < 0 \tag{3.7}$$

where terms of order ϵ^2 have been neglected. By successively chossing $d^* = b, b - \epsilon, b - 2\epsilon, \ldots,$ and repeatedly applying variation (3.4) until, say, $r = c_2$ where $M_r^2(c_2) = M_r^3(c_2)$, the moment volume will be successively reduced, provided that in the process $M_r^i (i = 1, 2)$ remain non-negative. It is assumed that this is the case. A typical sketch of the stresses thus obtained is shown in Fig. 4. Since in the remainder of the plate to be designed, i.e. $0 \le r \le c_2$, all of the stress states are isotropic, the process of reducing the volume may now be continued exactly as before where b is now replaced by c_2 and the superscripts 2 and 3 are interchanged. It is evident that the moment diagram depicted in Fig. 5 will be ultimately obtained. It may happen that the critical radius c_1 does not occur; then the loading p_1 does not effect the final design. Similarly if c_2 does not exist, the final design is independent of p_2.

It is evident that the foregoing technique can be just as easily applied for arbitrary n loads and q critical radii. The procedure was illustrated for $n = 3$ and $q = 2$ merely to facilitate clarity of Figs. 3–5. If n and q are considered to be arbitrary, it is clear that the final design must obey the following two rules:

Rule 1—Between any two consecutive critical radii, say c_m and c_{m+1}, the stress state will be at A (Fig. 1) for exactly one of the loadings; for each of the remaining loads, the stress state will either be AB or below yield.

Rule 2—If the stress state is below yield for a given load and over the subdomain $c_m \le r \le c_{m+1}$, the same stress state applies for that load everywhere in the region $0 \le r \le c_{m+1}$.

It follows from Rules 1 and 2 that if, for a given load, the stress state A does not apply anywhere, the final design will be independent of that load. It should again be noted that a design satisfying both rules is not always possible since it was tacitly assumed that during the process of varying the stresses none of the radial bending moments became negative. Assuming that the technique does produce a design satisfying both rules, it is not difficult to show that the design thus obtained satisfies (1.1) and therefore is indeed the absolute optimum; the proof is found in the appendix.

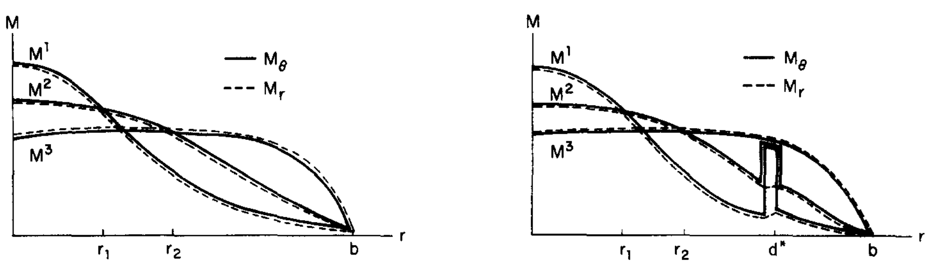

Fig. 3. Initial (a) and varied stress (b).

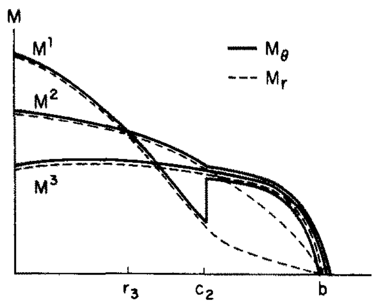

Fig. 4. Intermediate design.

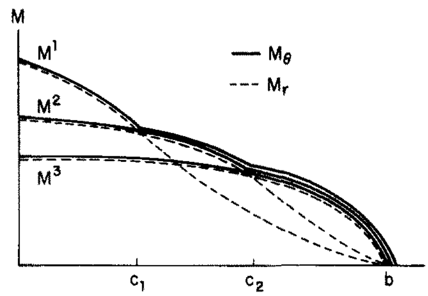

Fig. 5. Optimum design for three loads.

4. EXAMPLE

Consider a simply supported plate which is to carry separately each of three downward loads. Since the shear is statically determinate, the loads are completely specified by the three non-positive shear forces T_1, T_2 and T_3. Assuming all three shears affect the optimum design and that the minimal moments are of the form specified by Fig. 5, (2.1) may be integrated to yield

$$M_r^1=\begin{cases}\int_{c_1}^{r} T_1(\xi)\,d\xi+\int_{c_2}^{c_1} T_2(\xi)\,d\xi+\int_{b}^{c_2} T_3(\xi)\,d\xi & 0\le r\le c_1\\ r^{-1}\int_{b}^{r} \xi[T_1(\xi)-T_2(\xi)]\,d\xi+\int_{c_2}^{r} T_2(\xi)\,d\xi+\int_{b}^{c_2} T_3(\xi)\,d\xi & c_1\le r\le c_2\\ r^{-1}\int_{b}^{r} \xi[T_1(\xi)-T_3(\xi)]\,d\xi+\int_{b}^{r} T_3(\xi)\,d\xi & c_2\le r\le b\end{cases} \tag{4.1}$$

$$M_r^2=\begin{cases}\int_{c_2}^{r} T_2(\xi)\,d\xi+\int_{b}^{c_2} T_3(\xi)\,d\xi & 0\le r\le c_2\\ r^{-1}\int_{b}^{r} \xi[T_2(\xi)-T_3(\xi)]\,d\xi+\int_{b}^{r} T_3(\xi)\,d\xi & c_2\le r\le b\end{cases}$$

$$M_r^3=\left\{\int_{b}^{r} T_3(\xi)\,d\xi \qquad 0\le r\le b\right.$$

where the critical radii are the largest solution to

$$\int_{b}^{c_2} \xi[T_2(\xi)-T_3(\xi)]\,d\xi=0,$$
$$\int_{b}^{c_1} \xi[T_1(\xi)-T_2(\xi)]\,d\xi=0. \tag{4.2}$$

In the region $c_2\le r\le b$, it is clear that

$$M_r^3\ge M_r^2\ge M_r^1. \tag{4.3}$$

Substitution of the appropriate expressions from (4.1) into (4.3) yields

$$\int_{b}^{r} \xi[T_3(\xi)-T_2(\xi)]\,d\xi\ge 0$$

and

$$\int_{b}^{r} \xi[T_2(\xi)-T_1(\xi)]\,d\xi\ge 0. \tag{4.4}$$

Similarly, in the region $c_1\le r\le c_2$, the shear forces satisfy

$$\int_{c_2}^{r} [T_2(\xi) - T_3(\xi)]\, d\xi \geq 0,$$
$$\int_{b}^{r} \xi[T_2(\xi) - T_1(\xi)]\, d\xi \geq 0, \tag{4.5}$$

while for $0 \leq r \leq c_1$,

$$\int_{c_1}^{r} [T_1(\xi) - T_2(\xi)]\, d\xi \geq 0,$$
$$\int_{c_2}^{r} [T_2(\xi) - T_3(\xi)]\, d\xi \geq 0. \tag{4.6}$$

The necessary and sufficients conditions to insure a solution of the type being considered are: (a) there exist solutions c_1, c_2 to (4.2) such that $c_2 > c_1 > 0$; (b) $M_r^1(r) \geq 0$ for $c_1 \leq r \leq b$; (c) inequalities (4.4), (4.5) and (4.6) are satisfied.

Now consider the specific shears representing a concentrated force at the center, and two ring loads at the radii $b/3$ and $2b/3$. Thus the shear forces become

$$rT_1 = -F_1,$$
$$rT_2 = -F_2 H(r - b/3),$$
$$rT_3 = -F_3 H(r - 2b/3).$$

It may easily be observed that if c_1 and c_2 exist, then a necessary and sufficient condition for satisfaction of (4.4), (4.5) and (4.6) is

$$0 < F_1 < F_2 < F_3. \tag{4.7}$$

Substitution of (4.7) into (4.2) shows that

$$c_1 = b\left[1 - \frac{2F_2}{3F_1}\right]$$

and

$$c_2 = b\left[1 - \frac{F_3}{3F_2}\right]\dagger. \tag{4.8}$$

It may easily be shown that $c_1 \leq b/3$ and that $c_2 \leq 2b/3$. The conditions that $c_1 \geq 0$ and $c_2 \geq b/3$ yield the inequalities

$$F_3 \leq 2F_2 \leq 3F_1. \tag{4.9}$$

It remains to establish that $M_r^1 \geq 0$ for $c_1 \leq r \leq b$. First, consider the region $c_2 \leq r \leq b$, where

$$\frac{d^2}{dr^2}(rM_r^1) = T_3. \tag{4.10}$$

Since T_3 is negative for $r > 2b/3$ and zero for $r < 2b/3$, it follows that M_r^1 need only be evaluated at the radii $2b/3$ and c_2. Performing the calculations and using (4.8) and (4.9), it follows that

$$3rM_r^1|_{2b/3} = b(F_1 - 0{\cdot}189F_3) \geq 0{\cdot}433bF_1 > 0,$$

and (4.11)

†In deriving the expression for c_2, it was assumed that $c_2 \geq b/3$. If it was assumed that $c_2 \leq b/3$, the vanishing of both T_2 and T_3 for $0 \leq r \leq b/3$ would show that $c_2 = b/3$ is also a solution to (4.2). Since the largest solution c_2 is desired, it follows that $c_2 \geq b/3$ is correct.

$$3rM_r^1|_{c_2} = b\frac{F_3}{F_2}[-F_2+F_1+(3F_2-F_3)\ln 1{\cdot}5]$$
$$\geq 0{\cdot}216bF_3(F_2-F_1)/F_2 > 0.$$

Similarly, for $c_1 \leq r \leq c_2$, the radii $r = c_1$ and $r = b/3$ are the radii at which M_r^1 must be evaluated. It is not difficult to show that

$$M_r^1|_{b/3} = 2(F_1-F_2)+F_2\ln(3c_2/b)+F_3\ln 1{\cdot}5$$
$$\geq 0{\cdot}144F_2 > 0 \tag{4.12}$$

and

$$M_r^1|_{c_1} = F_2\ln(3c_2/b)+F_3\ln 1{\cdot}5 > 0.$$

It follows from (4.9) that if $2F_2 \leq F_3 \leq 3F_1$, then a non-negative solution c_1 still exists but c_2 does not exist. Moreover, if $F_3 \leq 2F_2$ and $2F_2 \geq 3F_1$, then c_2 exists but c_1 does not. This suggests that the loading F_2 does not affect the design in the former case, while F_1 does not affect the design in the latter case. That this is the proper interpretation shall now be established.

The optimum design, which is unaffected by F_2, is obtained by deleting M_r^2 and M_θ^2 in Fig. 5. Therefore the design M_0 is given by

$$\left.\begin{aligned} &M_0 = M_r^1 = M_\theta^1 \\ &0 \leq M_r^3 = M_\theta^3 \leq M_0, \end{aligned}\right\} \quad 0 \leq r \leq c_1$$
$$\left.\begin{aligned} &M_0 = M_r^3 = M_\theta^3 = M_\theta^1 \\ &0 \leq M_r^1 \leq M_0, \end{aligned}\right\} \quad c_1 \leq r \leq b \tag{4.13}$$

and the stresses are depicted in Fig. 6. An analysis similar to the previous case yields:

$$\int_b^{c_1} \xi[T_1(\xi)-T_3(\xi)]\,d\xi = 0, \tag{4.14}$$

from which

$$c_1 = b\left[1-\frac{F_3}{3F_1}\right] \tag{4.15}$$

is obtained. The condition $c_1 > 0$ implies that

$$F_3 < 3F_1. \tag{4.16}$$

With (4.15) and (4.16), it is a straightforward procedure to verify the inequalities in (4.13).

Next, the existence of a statically admissible M_r^2, M_θ^2 nowhere violating the yield condition must be established. Choose

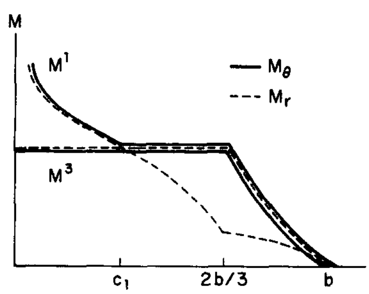

Fig. 6. Optimum design for F_1 and F_3.

$$M_\theta^2 = M_r^3 \qquad b/3 \le r \le b$$
$$M_\theta^2 = M_r^2 \qquad 0 \le r \le b/3\dagger. \tag{4.17}$$

From equilibrium and (4.17), it follows that

$$M_r^2 = \begin{cases} \int_b^r T_3(\xi)\,d\xi + r^{-1}\int_b^r \xi[T_2(\xi) - T_3(\xi)]\,d\xi & b/3 \le r \le b \\ \int_b^{b/3} T_3(\xi)\,d\xi + (b/3)^{-1}\int_b^{b/3} \xi[T_2(\xi) - T_3(\xi)]\,d\xi & 0 \le r \le b/3. \end{cases} \tag{4.18}$$

Observing that $M_\theta^2 = M_\theta^3$ for $r \ge b/3$, it follows that

$$r(M_r^3 - M_r^2) = \int_b^r \xi[T_3(\xi) - T_2(\xi)]\,d\xi \ge 0, \tag{4.19}$$

where the inequality is a consequence of the assumption that the stresses corresponding to F_2 are at or below yield. Evaluation of the integral in (4.19) yields:

$$2F_2 \le F_3. \tag{4.20}$$

That M_r^2 is non-negative for $b/3 \le r \le b$ may be established by evaluating M_r^2 at $b/3$ and $2b/3$. Performing the calculations shows that

$$2M_r^2|_{2b/3} = 2F_3 \ln 1{\cdot}5 + F_2 - F_3 > 0$$
$$M_r^2|_{b/3} = F_3 \ln 1{\cdot}5 + 2F_2 - F_3 > 0, \tag{4.21}$$

where the inequality signs in (4.21) follow from (4.7), (4.16) and (4.20).

Now M_r^2 is non-negative in the central region $0 \le r \le b/3$. It remains to show that M_r^2 is at or below yield in that region. If $c_1 > b/3$, (4.13) and (4.19) are used to show that $M_r^1 \ge M_r^3 \ge M_r^2$ for $r \le b/3$. If $c_1 < b/3$, it suffices to show that $M_0 = M_r^3 \ge M_r^2$ for $c_1 \le r \le b/3$ and that $M_0 = M_r^1 \ge M_r^2$ for $r \le c_1$. Routine calculations show that

$$M_r^3 - M_r^2 = F_3 - 2F_2 \ge 0 \qquad \text{for} \qquad c_1 \le r \le b/3$$
$$M_r^1 - M_r^2 \ge M_r^1(c_1) - M_r^2(c_1) = F_3 - 2F_3 \ge 0 \qquad \text{for} \qquad r \le c_1. \tag{4.22}$$

Now consider the case where c_1 does not exist, so that the design is assumed to be independent of F_1. The form of the design is obtained by deleting the curves for M_r^1 and M_θ^1 in Fig. 5. Omitting details,

$$F_3 \le 2F_2 \tag{4.23}$$

results from the condition $c_2 \ge b/3$. Again, the admissibility of the stresses can be easily established.

To establish the existence of M_r^1, M_θ^1 at or below yield, special care is needed to insure the $|M_r^1| < \infty$ at the location of the centrally applied concentrated force. Consider

$$M_\theta^1 = M_\theta^2 \qquad r^* \le r \le b$$
$$M_\theta^1 = F_1 \qquad 0 \le r \le r^*, \tag{4.24}$$

†This choice is a natural consequence of the method discussed in Section 3. Successive application of the stress variation (3.6) eventually leads to the form (4.17). There is no value in further applying the variation to M_r^2, M_θ^2 in the range $0 \le r \le b/3$ since the minimum value of c_2 is $b/3$, i.e. c_2 does not exist.

where r^* is defined by the greatest solution, other than b, to

$$M_r^1(r^*) = 0. \tag{4.25}$$

Assuming $r^* < b/3$, it can easily be shown that

$$r^*F_2 \ln(3c_2/b) + r^*F_3 \ln 1{\cdot}5 + F_1(b - r^*) - 2F_2b/3 = 0. \tag{4.26}$$

The condition $M_r^2 \geq M_r^1$ for $r^* \leq r \leq b$ leads to

$$2F_2 \geq 3F_1(1 - r^*/b). \tag{4.27}$$

It remains to show that $M_r^2(r^*) \geq F_1$. Now

$$\begin{aligned} M_r^2(r^*) - F_1 &= F_2 \ln(3c_2/b) + F_3 \ln 1{\cdot}5 - F_1 \\ &= \frac{b}{3r^*}(2F_2 - 3F_1), \end{aligned} \tag{4.28}$$

where (4.26) was used to establish the latter result in (4.28). Therefore F_1 does not affect the design if

$$2F_2 > 3F_1 \tag{4.29}$$

as originally anticipated. It should be pointed out that (4.29) was derived by assuming $r^* < b/3$. Recalling the effect of the stress variation of the previous section, it is obvious that r^* increases as F_1 decreases. Therefore the lower bound theorem of limit analysis shows that if $r^* > b/3$, the stresses corresponding to the load F_1 must still be at or below yield.

5. DISCUSSION

It has been shown that the statical method of stress variation can systematically lead to designs which may be tested for optimality by (1.1). The method has been illustrated for a simply supported plate for an arbitrary number of separate loads, each of which must be safely carried by the plate. The single stress variation (3.4) is sufficient to obtain the optimum design provided that in the process none of the radial bending moments become negative.

The essential feature of variation (3.4) is that is applicable whenever the statical boundary conditions are $M_r = 0$ at the outer edge, and M_r unspecified at the inner edge. Therefore, all that was developed specifically for the supported plate is equally applicable to annular plates either clamped at $r = a$, or supported at $r = b$ but with a rigid central boss constraining the inner edge against rotation.

In most practical cases where either the number n of separate loadings is large or where the shear forces are not of a very simple form, analytic solutions are not feasible. However, it is a very simple matter to program the method for modern high speed computers. In contrast to linear programming techniques[10] where the coefficient matrix is proportional to n, in the present method it is the computational time that is proportional to n. Additional time savings may be achieved by discarding any stresses once it is determined that they cannot affect the final design.

Of course, for a great many cases this method will fail as one or more of the radial moments becomes negative. This case has never been previously treated in the literature. In fact, the examples of [8, 9] assume that the side AB and corner A alone determine the optimum. This technique could be very easily extended to show that if M_r^i goes negative for some i, then variation (3.4) is still applicable provided that now the constant ξ_i is set equal to M_0. Then the stresses (M_r^i, M_θ^i) will lie on side BC of the Tresca condition. However, much less trivial difficulties develop if M_θ^i now goes negative. Moreover, as the minimal stresses increase in complexity, the proof that the final design obtained, when it its most general form, satisfies (1.1) becomes increasingly difficult. For this reason, work is in progress to refine the stress variation technique so as to make it independent of (1.1). Results will be reported in the near future for more general loads and for other kinematic support conditions.

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APPENDIX

Derivation of the optimality condition

Take the first variation of (2.5) and use (3.3) to obtain

$$\delta J = \sum_{i=1}^{n} \int_a^b \left[\frac{\partial M_0}{\partial M_r^i}\delta M_r^i + \frac{\partial M_0}{\partial M_\theta^i}\frac{\mathrm{d}}{\mathrm{d}r}(r\delta M_r^i)\right] r\,\mathrm{d}r$$

$$- \sum_{i=1}^{n}\sum_{j=1}^{k} \delta R_j^i \left\{\int_a^b \frac{\partial M_0}{\partial M_\theta^i} d_j H(r-d_j) r\,\mathrm{d}r - B\right\}, \qquad \text{(A1)}$$

where B is a Lagrangian multiplier. Integration of the latter term of the first integral in (A1) by parts and the requirement that δJ vanish for all variations δM_r^i and δR_j^i yields the Euler conditions for an extremal

$$\frac{\partial M_0}{\partial M_r^i} - \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\partial M_0}{\partial M_\theta^i}\right) = 0 \qquad (i = 1, \ldots, n) \qquad \text{(A2)}$$

and

$$\int_{d_j}^{b} r\frac{\partial M_0}{\partial M_\theta^i}\,\mathrm{d}r = B \qquad (i = 1, \ldots, n). \qquad \text{(A3)}$$

Equation (A3) may also be expressed by

$$\int_{d_j}^{d_{j+1}} r\frac{\partial M_0}{\partial M^i}\,\mathrm{d}r = 0, \qquad (i = 1, \ldots, n), (j = 1, \ldots, k-1). \qquad \text{(A4)}$$

The natural boundary conditions associated with (A2) and (A4) are

$$r\frac{\partial M_0}{\partial M_\theta^i} = 0, \qquad (i = 1, \ldots, n) \qquad \text{(A5)}$$

at any edge for which M_r^i is not prescribed. Sufficiency of the extremal conditions follows from the convexity and homogeneity of order one for the function $M_0(M_r^i, M_\theta^i)$[11].

It remains to derive (1.1) from (A2), (A4) and (A5). In (2.3), if the stresses are ordered so that

$$M_0^i \begin{cases} = M_0, & (i = 1, \ldots, p) \\ < M_0, & (i = p+1, \ldots, n), \end{cases} \qquad \text{(A6)}$$

(A2) takes the form

$$\frac{\partial M_0}{\partial M_0^i}\frac{\partial M_0^i}{\partial M_r^i} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\partial M_0}{\partial M_0^i}\frac{\partial M_0^i}{\partial M_\theta^i}\right), \qquad (i = 1, \ldots, p)$$

$$\frac{\partial M_0}{\partial M_r^i} = \frac{\partial M_0}{\partial M_\theta^i} = 0, \qquad (i = p+1, \ldots, n). \qquad \text{(A7)}$$

The derivatives $\partial M_0/\partial M_0^i$ are the components of the gradient to the hypersurface $M_0 = M_0(M_0^1, M_0^2, \ldots, M_0^p)$ at the corner $M_0^1 = M_0^2 = \cdots = M_0^p$. The case $p = 2$ is illustrated in Fig. 7. Although the components $\partial M_0/\partial M_0^i$ of the gradient are not unique, they are not totally arbitrary either. Since M_0 is a homogeneous function of order one in the argument M_0^i, it follows that

$$M_0 = \sum_{i=1}^{p} M_0^i \frac{\partial M_0}{\partial M_0^i} \qquad \text{(A8)}$$

and since $M_0 = M_0^1 = \cdots = M_0^p$, that

$$\sum_{i=1}^{p} \frac{\partial M_0}{\partial M_0^i} = 1. \qquad \text{(A9)}$$

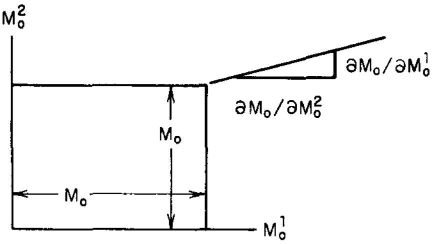

Fig. 7. Hypersurface for $M_0(M_0^i)$.

According to the plastic potential flow law, the incipient curvature rates κ_r^i, κ_θ^i at plastic collapse satisfy

$$\begin{aligned}\kappa_r^i &= \lambda^i \partial M_0^i/\partial M_r^i \\ \kappa_\theta^i &= \lambda^i \partial M_0^i/\partial M_\theta^i,\end{aligned} \qquad (i = 1, \dots, p) \tag{A10}$$

where λ^i are non-negative scalars. Substitution of (A10) into (A7) results in

$$\frac{d}{dr}\left(\frac{1}{\lambda^i}\frac{\partial M_0}{\partial M_0^i}\right) = 0, \qquad (i = 1, \dots, p), \tag{A11}$$

for which the solution is

$$\frac{\partial M_0}{\partial M_0^i} = \lambda^i, \tag{A12}$$

where the non-essential positive constant of integration has been set equal to unity. The energy dissipation D_i resulting from the plastic response to the load p_i is

$$D_i = M_r^i \kappa_r^i + M_\theta^i \kappa_\theta^i. \tag{A13}$$

With the use of (A10) and (A12) and Euler's theorem of homogeneous functions, (A12) becomes

$$D_i = M_0^i \frac{\partial M_0}{\partial M_0^i}. \tag{A14}$$

Equation (A8) and (A14) now easily result in (1.1).

Since (A10) and (A12) imply that $\kappa_\theta^i = \partial M_0/\partial M_\theta^i$, and recalling that $r\kappa_\theta^i = -dw^i/dr$ where w^i is the collapse velocity, (A4) becomes merely the kinematic condition $w(d_j) = w(d_{j+1})$ and (A5) becomes the kinematic boundary condition at any edge constrained against rotation, i.e. where M_r is not prescribed.

Proof of optimality for the rules

Here, it will be established that if the stress variation technique produces a design satisfying Rules 1 and 2, then that design is optimal. Denote the q critical radii by $c_l(l = 1, \dots, q)$. For $0 \le r \le c_1$, one of the stresses is at point A, the remainder below yield. Assuming $M_r^n = M_\theta^n = M_0$, (1.1) becomes $\kappa_r^n + \kappa_\theta^n = 1$. Use of the compatibility conditions $d(r\kappa_\theta^n)/dr = \kappa_r^n$ and the boundedness requirement at the origin yields $\kappa_r^n = \kappa_\theta^n = 1/2 > 0$ which is compatible with flow law requirements at point A.

The proof is completed by appealing to the principle of mathematical induction, i.e. assume that the solution is kinematically admissible for $c_{\alpha-1} \le r \le c_\alpha$, from which admissibility for $c_\alpha \le r \le c_{\alpha+1}$ must be shown. For definiteness, assume $M_\theta^1 = M_r^1 = M_0$ for the former region and $M_\theta^2 = M_r^2 = M_0$ for the latter region. Then for $c_{\alpha-1} \le r \le c_\alpha$,

$$\begin{aligned}\kappa_r^i &= 0 \\ \kappa_\theta^i &= A_i/r\end{aligned} \qquad (i = 2, \dots, n) \tag{A15}$$

where A_i are non-negative constants. Substitution of (A15) and the isotropic condition $M_\theta^1 = M_r^1 = M_0$ into (1.1) yields

$$\kappa_r^1 + \kappa_\theta^1 = 1 - \sum_{i=2}^{n} A_i/r. \tag{A16}$$

Furthermore, substitution of (A16) into the compatibility condition for $(\kappa_r^1, \kappa_\theta^1)$ produces

$$\begin{aligned} r^2\kappa_\theta^1 &= \frac{1}{2}(r^2 - c_{\alpha-1}^2) + c_{\alpha-1}^2 k^2 + \sum_{i=2}^{n} A_i(c_{\alpha-1} - r) \\ r^2\kappa_r^1 &= \frac{1}{2}(r^2 + c_{\alpha-1}^2) - c_{\alpha-1}^2 k^2 - c_{\alpha-1}\sum_{i=2}^{n} A_i, \end{aligned} \tag{A17}$$

where $k^2 = \kappa_\theta^1(c_{\alpha-1}) \ge 0$ according to the hypothesis.

Now assume that $M_r^2 = M_\theta^2 = M_0$ for $c_\alpha \le r \le c_{\alpha+1}$, where the remainder of the stresses lie on side AB. The equations, analogous to (A15) and (A17), subject to continuity of $\kappa_\theta^i (i = 1, \dots, n)$ at c_α are

$$\begin{aligned}\kappa_r^i &= 0 \\ \kappa_\theta^i &= A_i/r > 0 \quad (i = 3, \dots, n)\end{aligned} \tag{A18}$$

$$\kappa_\theta{}^1 = c_\alpha r^{-1} \kappa_\theta{}^1(c_\alpha) > 0$$
$$\kappa_r{}^1 = 0 \tag{A19}$$

$$r^2 \kappa_r{}^2 = \frac{1}{2}(r^2 + c^2_{\alpha-1}) - c^2_{\alpha-1} k^2 - c_{\alpha-1} \sum_{i=2}^{n} A_i$$

$$r^2 \kappa_\theta{}^2 = \frac{1}{2}(r^2 - c^2_{\alpha-1}) + c^2_{\alpha-1} k^2 + A_2 c_{\alpha-1} + \sum_{i=3}^{n} A_i (c_{\alpha-1} - r) - c_\alpha r \kappa_\theta{}^1(c_\alpha) \tag{A20}$$

Since $\mathrm{d}(r\kappa_\theta{}^2)/\mathrm{d}r = \kappa_r{}^2$ and $\kappa_\theta{}^2(c_\alpha) > 0$, both curvature rates will be non-negative and therefore admissible if $\kappa_r{}^2 \geq 0$. But

$$\kappa_r{}^2 \geq \frac{c_\alpha{}^2}{r^2} \kappa_r{}^1(c_\alpha) \geq 0. \tag{A21}$$

Since the assumed design is compatible with (1.1) for $c_\alpha \leq r \leq c_{\alpha+1}$, it follows that it is also compatible with (1.1) for $0 \leq r \leq b$; consequently it is optimal.